Last time: Connected geometry with algebra (for vectors)

12.3: Dot Product

Goal: Connect algebra of vectors to their geometry via a new operation on vectors

Definition: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$

The dot product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

(vector • vector $\mapsto$ scalar)

Q: Does $\vec{u} \cdot (\vec{v} \cdot \vec{w})$ make sense?

No! It doesn't make sense. $\vec{v} \cdot \vec{w}$ makes a scalar so this problem tries to take a vector dot a scalar.

($(\vec{u} \cdot \vec{v})\vec{w}$) makes sense via scalar multiplication)

Ex: $\vec{u} = \langle 1, 3, 5 \rangle$, $\vec{v} = \langle -3, 5, 7 \rangle$

$$\vec{u} \cdot \vec{v} = (1)(-3) + (3)(5) + (5)(7)$$

$$= -3 + 15 + 35 = \underline{47}$$

Theorem (Properties of dot product): (POD)

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

1) $\vec{v} \cdot \vec{v} = v_1 v_1 + v_2 v_2 + v_3 v_3 = v_1^2 + v_2^2 + v_3^2 = |\vec{v}|^2$

2) $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \vec{v} \cdot \vec{u}$

3) $\vec{u} \cdot (\vec{v} + \vec{w}) = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$

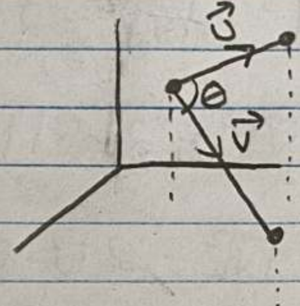
$$= u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3)$$

$$= (u_1 v_1 + u_2 v_2 + u_3 v_3) + (u_1 w_1 + u_2 w_2 + u_3 w_3) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

4) $\underline{\vec{u}\cdot(c\vec{v})} = \langle u_1, u_2, u_3\rangle \cdot \langle cv_1, cv_2, cv_3\rangle$
$= u_1(cv_1) + u_2(cv_2) + u_3(cv_3)$
$= c(u_1v_1 + u_2v_2 + u_3v_3) = c(\vec{u}\cdot\vec{v})$

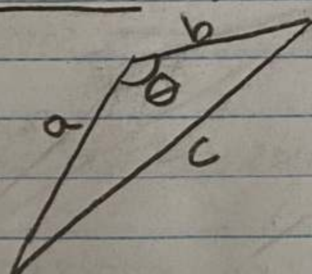
5) $\vec{0}\cdot\vec{v} = 0$ "zero property"

<u>Theorem (Geometric interpretation of Dot Product):</u>
Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and let θ be the angle between them



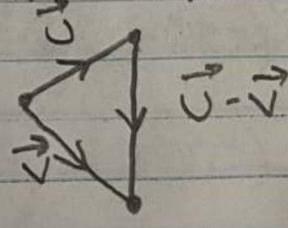
Then $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$
purely algebraic purely geometric

<u>Recall</u>: Law of cosines



$c^2 = a^2 + b^2 - 2ab\cos(\theta)$
(generalization of the Pythagorean Theorem)

<u>Proof:</u> Consider $\vec{u}-\vec{v}$. Applying law of cosines
$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$



On the left side, we apply properties of the dot product.

$$
\begin{aligned}
|\vec{u}-\vec{v}|^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) && \text{(Part 1 POD)}\\
&= (\vec{u}-\vec{v})\cdot\vec{u} - (\vec{u}-\vec{v})\vec{v} && \text{(Part 2 POD)}\\
&= (\vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u}) - (\vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{v}) && \text{(Distribution of dot)}\\
&= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} && \text{(Algebra)}\\
&= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\vec{u}\cdot\vec{v} && \text{(communitivity of dot)}\\
&= |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\cdot\vec{v} && \text{(Part 1 again)}
\end{aligned}
$$

Hence $|\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta)$

$= |\vec{u}-\vec{v}|^2$

$= |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\cdot\vec{v}$

Finally, $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$

Corollary: Supposing $\vec{u},\vec{v}$ are both nonzero

$$\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

* arccos same as $\cos^{-1}$

Observation: The zero-vector has an undefined angle with all other vectors

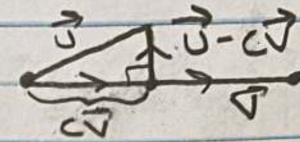
So having an angle means no non-zero vectors

Corollary: If $\vec{u}$ and $\vec{v}$ are perpendicular (i.e. orthogonal), then $\vec{u}\cdot\vec{v}=0$

Conversely, $\vec{u}\cdot\vec{v}=0$ implies $\vec{u}$ and $\vec{v}$ are orthogonal

Orthogonal Projection

Suppose $\vec{u}, \vec{v} \in \mathbb{R}^n$



To project $\vec{u}$ orthogonally onto $\vec{v}$:

$$c\vec{v} \cdot (\vec{u} - c\vec{v}) = 0$$

if $$c(\vec{v} \cdot \vec{u}) - c^2(\vec{v} \cdot \vec{v}) = 0$$

$$c(\vec{v} \cdot \vec{u} - c|\vec{v}|^2) = 0$$

either $c = 0$ or $\vec{u} \cdot \vec{v} - c|\vec{v}|^2 = 0$

So assuming $|\vec{v}| \neq 0$ and $c \neq 0$

$$c = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}$$

Definition: The orthogonal projection of $\vec{u}$ onto $\vec{v}$ is:

$$\mathrm{Proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\vec{v}$$

$$= \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \cdot \left(\frac{1}{|\vec{v}|}\vec{v}\right)$$

$$= \mathrm{Comp}_{\vec{v}}(\vec{u}) \left(\frac{1}{|\vec{v}|}\vec{v}\right)$$

The scalar projection of $\vec{u}$ onto $\vec{v}$ is $\mathrm{Comp}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$

Direction Angles

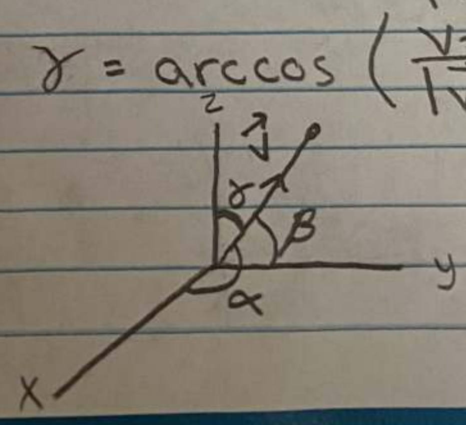
Let $\vec{v} \in \mathbb{R}^3$ $\qquad \vec{v} = \langle v_1, v_2, v_3 \rangle$

The direction angles of $\vec{v}$ are the angles $\vec{v}$ makes with $\vec{i}$, $\vec{j}$, and $\vec{k}$.

i.e. $$\alpha = \arccos\left(\frac{\vec{v} \cdot \vec{i}}{|\vec{v}||\vec{i}|}\right) = \arccos\left(\frac{v_1}{|\vec{v}|}\right)$$

$$\beta = \arccos\left(\frac{v_2}{|\vec{v}|}\right)$$

$$\gamma = \arccos\left(\frac{v_3}{|\vec{v}|}\right)$$



The direction angles determine the "would-be location" of $\vec{v}$ on the unit sphere about the origin

Exercise: Show that any two of the direction angles of $\vec{v}$ determine the third...